Received: *December 15, 2000,* Accepted: *February 27, 2001*
Revised: *February 19, 2001*

Brane-antibrane systems on Calabi-Yau spaces

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Abstract: We propose a correspondence between brane-antibrane systems and stable triples (E_1, E_2, T), where E_1, E_2 are holomorphic vector bundles and the tachyon T is a map between them. We demonstrate that, under the assumption of holomorphicity, the brane-antibrane field equations reduce to a set of vortex equations, which are equivalent to the mathematical notion of stability of the triple. We discuss some examples and show that the theory of stable triples suggests a new notion of BPS bound states and stability, and curious relations between brane-antibrane configurations and wrapped branes in higher dimensions.

Keywords: D-branes, Differential and Algebraic Geometry.

JHEP02(2001)045

Contents

1. Introduction

Systems of non-BPS brane configurations have been extensively studied recently (for a review see [1]). A basic non-BPS system is the coincident brane-antibrane configuration, which is not stable. It has a tachyon on the world-volume of the branes that arises from the open string stretched between the branes and the antibranes, and it is charged under the world-volume gauge groups. The decay of the system can be seen, from various viewpoints [2, 13], by the tachyon rolling down to the minimum of its potential. Upon tachyon condensation one can end up with lower dimensional BPS branes, if the original brane-antibrane system contained the corresponding charges.

In another line of research, much progress has been achieved in the study of BPS branes at arbitrary points in the moduli space of type-II string theory compactified on Calabi-Yau spaces [14, 26]. In particular a new concept of stability has been introduced [22].

This work will relate to these two lines of research. We will propose a correspondence between brane-antibrane systems and stable triples (E_1, E_2, T), where

E_1, E_2 are holomorphic vector bundles and the tachyon, T, is a map between them. We demonstrate that, under the assumption of holomorphicity, the brane-antibrane field equations reduce to a set of vortex equations. The latter are equivalent to the topological notion of stability of the triple (E_1, E_2, T). This is quite analogous to the case of a single vector bundle where solutions of hermitian Yang-Mills equations correspond to stable holomorphic bundles. We discuss some examples and show that the theory of stable triples suggests a new notion of BPS bound states upon tachyon condensation, and curious relations between brane-antibrane configurations and wrapped branes in higher dimensions.

This paper is organized as follows. In section 2 we will propose the correspondence, and show the equivalence between the (holomorphic) field equations of the brane-antibrane system and the vortex equations. We will show that such configurations saturate a Bogomol'nyi bound and demonstrate the relation by some examples. In section 3 we will relate our description of BPS D-branes as stable triples to existing descriptions of BPS states. In particular, we will show that the theory of stable triples suggests a new notion of BPS bound states and stability. This may also allow new BPS configurations that arise upon tachyon condensation. In section 4 we discuss some generalizations, and will see that the theory of stable triples suggests some curious relations between brane-antibrane configurations and wrapped branes in higher dimensions.

2. Tachyon condensation and stable triples

In this section we will propose a correspondence between systems of coinciding branes and antibranes wrapping a manifold X and stable triples (E_1, E_2, T). E_1 and E_2 are holomorphic vector bundles on X. Physically, they correspond to the branes and antibranes respectively. T is a homomorphism between the vector bundles $T : E_2 \to E_1$. It is the tachyon field that arises from the open string stretched between the branes and the antibranes.[1] With a particular holomorphic ansatz, we will recast the field equations of the brane-antibrane system as a set of vortex equations. We argue that solutions of the vortex equations represent BPS configurations. Such solutions correspond to a mathematical construction of stable triples on X [28]. With this correspondence the analysis of tachyon condensation leading to BPS branes will be replaced by a stability analysis.

2.1 Tachyon condensation and triples

Let us now briefly review the construction of D-branes from coincident brane-antibrane configurations [3]. We will consider the particular case of type-IIA string theory compactified on Calabi-Yau manifold X of complex dimension d.

[1] To match the math literature we think of T as a map from antibranes to branes. The conjugate field $\bar{T}$ is the map in the other direction.

A configuration of n branes wrapping the Calabi-Yau manifold is described by a $\mathrm{U}(n)$ vector bundle E on X. This carries charges for various RR fields which, as shown in [29, 30], takes the form

$$Q = \mathrm{ch}(E)\sqrt{\widehat{\mathrm{A}}(X)} = \mathrm{ch}(E)\sqrt{\mathrm{Td}(X)}, \tag{2.1}$$

which is an element of the cohomology $H^*(X, \mathbb{Z})$ known as the Mukai vector. For the equality in (2.1) we used the fact that $\mathrm{Td}(X)$ on a Calabi-Yau manifold is equal to the A-roof genus $\widehat{\mathrm{A}}(X)$. In this expression $\mathrm{ch}(E)$ is the Chern character of the vector bundle E

$$\mathrm{ch}(E) = \mathrm{Tr}\exp\left[\frac{F}{2\pi}\right], \tag{2.2}$$

where F is the field strength of the gauge field on the brane. It has an expansion in terms of the Chern classes

$$\mathrm{ch}(E) = n + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + \cdots . \tag{2.3}$$

The A-roof genus $\widehat{\mathrm{A}}(X)$ and has an expansion in terms of the Pontrjagin classes

$$\widehat{\mathrm{A}}(X) = 1 - \frac{p_1(X)}{24} + \cdots . \tag{2.4}$$

Now suppose we have a configuration of n_1 branes wrapped on X, together with n_2 antibranes. In general, the configuration is described by specifying a $\mathrm{U}(n_1)$ vector bundle E_1 on X for the branes together with a second $\mathrm{U}(n_2)$ bundle E_2 for the antibranes. The net D-brane charge is then the difference of the Mukai vectors for the two bundles

$$Q = Q_1 - Q_2 = (\mathrm{ch}(E_1) - \mathrm{ch}(E_2))\sqrt{\widehat{\mathrm{A}}(X)}. \tag{2.5}$$

In general, we would expect the antibranes to annihilate against the branes. However, if the bundles are different then there is a net D-brane charge so the branes cannot completely annihilate and still conserve charge.

Since identical bundles can annihilate, adding the same bundle to E_1 and E_2 gives the same physical configuration. That is we should identify $(E_1 \oplus V, E_2 \oplus V)$ with (E_1, E_2), which is the equivalence class identification made in K-theory. In K-theory one identifies the Chern class of the pair (E_1, E_2) with the difference of Chern classes $\mathrm{ch}(E_1) - \mathrm{ch}(E_2)$, thus providing the map from K-theory to the Mukai charge (2.5). In fact, a D-brane charge is more accurately measured by the K-theory class rather than the cohomological Mukai charge. In particular, K-theory includes more information than the Chern classes themselves. For instance, Chern classes miss torsion.

Physically, the annihilation happens because there is a tachyonic mode T in the open string connecting the branes and antibranes. The tachyon potential has a minimum away from zero. However, if the bundles E_1 and E_2 are different, there is a topological obstruction to the tachyon being at the minimum everywhere on X. The tachyon T transforms in the fundamental representation of each bundle $(\mathbf{n_2}, \bar{\mathbf{n}}_\mathbf{1})$, and it must respect the twisting of each bundle. In general, even if $n_1 = n_2$ if the bundles are different, it cannot do so and remain everywhere at the minimum of the tachyon potential. Instead it must be zero on some sub-manifold C of X. There is a vortex solution representing a lower-dimensional brane localized on C. In particular, all the lower dimensional branes can be built out of D9-branes in this way [3].

To specify a general D-brane configuration we need to specify the bundles E_1 and E_2 together with the condensed tachyon T. Since T is in the bi-fundamental representation, it represents a map between the bundles. Thus the full information is the triple (E_1, E_2, T) giving the complex

$$E_2 \xrightarrow{T} E_1 \, . \tag{2.6}$$

As we have noted, the D-brane charges are characterized by the K-theory class of the pair (E_1, E_2). However for a given D-brane charge there is generically a moduli space of different D-brane states. It is natural to ask what characterizes these distinct D-brane states. In general, this should be some equivalence class of triples (E_1, E_2, T), giving a finer classification than simply the K-theory class. In this paper, we will consider BPS configurations. We will see that this implies that the bundles and maps are holomorphic. A possible equivalence class has then been proposed in [31]. The suggestion is that for holomorphic bundles E_1 and E_2, we should identify triples in the same derived category [32, 33], which essentially means considering complexes of bundles of the form (2.6), modulo exact sequences. Note, however, that such an identification is not suitable for measuring stability, as the notion of stability is intrinsic only for abelian categories, while the derived category is just additive.

2.2 The vortex equations

Consider the low-energy effective action of the world-volume theory of a configuration of coinciding n_1 branes and n_2 antibranes wrapping a manifold X

$$S = \int_X \left[\frac{1}{4} \operatorname{Tr}_1 F_1^2 + \frac{1}{4} \operatorname{Tr}_2 F_2^2 + (DT)^a{}_{\bar{b}} (DT^*)^{\bar{b}}{}_a + \lambda \left(T^a{}_{\bar{b}} T^{*\bar{b}}{}_b - \alpha^2 \delta^a{}_b \right)^2 \right] . \tag{2.7}$$

Typically, we will take $n_1 = n_2$, so only lower-dimensional branes remain after condensation. There are higher order corrections to (2.7), and in general one also expects, as for the non-BPS branes [34], the kinetic terms of the tachyon and the gauge fields to depend on the tachyon background. Moreover, massive string states are expected to participate in the condensation process too. Such corrections modify the field equations and the precise description of the tachyon rolling to the minimum of its potential [35]. We expect and assume, however, that it should not matter for the topological construction of the lower-dimensional branes upon the condensation of the tachyon. Here we think about the lower-dimensional branes as the BPS branes or the stable non-BPS ones (where the tachyonic mode is projected out). Quantitative properties of the lower-dimensional branes such as the size of vortex solutions will be modified, upon the inclusion of the corrections.

We further assume the same gauge coupling for the two gauge groups and rescaled the gauge and tachyon fields in (2.7). In (2.7), a is the index of the fundamental representation of E_1 and $\bar{a}$ the anti-fundamental of E_2. The parameter α^2 in the tachyon potential is related to the valuse of the tachyon field at the minimum of the potential

$$\alpha^2 = \frac{1}{n_1} \operatorname{Tr}(TT^*)|_{\text{minimum}} . \tag{2.8}$$

Since T has charge $\pm e$ under the gauge groups and its covariant derivative is

$$D_M T^a_{\ \bar{b}} = \partial_M T^a_{\ \bar{b}} + ie(A^1_M)^a_{\ b} T^b_{\ \bar{b}} - ieT^a_{\ \bar{a}}(A^2_M)^{\bar{a}}_{\ \bar{b}} . \tag{2.9}$$

From now on we will suppress the indices and write, for instance $DT = dT + ieA_1 T - ieTA_2$.

The equations of motion read

$$\begin{aligned} D_1^M F^1_{MN} &= ie\left[T\left(D_N T^*\right) - \left(D_N T\right) T^*\right] \\ D_2^M F^2_{MN} &= ie\left[T^*\left(D_N T\right) - \left(D_N T^*\right) T\right] \\ D^2 T &= 2\lambda\left(TT^*T - \alpha^2 T\right) , \end{aligned} \tag{2.10}$$

where $D_1 = d + ieA_1$ and $D_2 = d + ieA_2$.

We denote the Kähler metric on X by $g_{m\bar{n}}$, where m is a holomorphic index and $\bar{n}$ an anti-holomorphic index. There is then a set of equations which imply the equations of motion. They are, first, that all the fields are holomorphic, namely

$$\begin{aligned} F^1_{mn} &= F^1_{\bar{m}\bar{n}} = 0 \\ F^2_{mn} &= F^2_{\bar{m}\bar{n}} = 0 \\ D_{\bar{m}} T &= 0 . \end{aligned} \tag{2.11}$$

Then in addition we have a hermitian condition

$$\begin{aligned} ig^{m\bar{n}} F^1_{m\bar{n}} + eTT^* &= 2\pi\tau_1 I_1 \\ ig^{m\bar{n}} F^2_{m\bar{n}} - eT^*T &= 2\pi\tau_2 I_2 , \end{aligned} \tag{2.12}$$

where I_1, I_2 are the identity matrices for the E_1 and E_2 bundles respectively. Together we shall call equations (2.11) and (2.12) the *vortex equations*. These equations were originally introduced and studied in [27], where it is also shown that their solutions arise as minima of the functional S. An important property, which we discuss in the next subsection, is that solutions of the vortex equations (2.11) and (2.12) are in one-to-one correspondence with the topological notion of stability of the triple (E_1, E_2, T) [27, 28].

To see[2] that the vortex equations do indeed imply the equations of motion (2.10), consider first the consequences of the holomorphic conditions (2.11). Recall that the Bianchi identity on either F^1 or F^2 reads

$$D_m F_{n\bar{n}} + D_n F_{\bar{n}m} + D_{\bar{n}} F_{mn} = 0\,. \tag{2.13}$$

Using the holomorphicity condition and contracting with $g^{m\bar{n}}$ implies that

$$\begin{aligned} D^M F_{Mn} &= g^{m\bar{n}} D_m F_{\bar{n}n} \\ &= -D_n \left(g^{m\bar{n}} F_{m\bar{n}}\right). \end{aligned} \tag{2.14}$$

Since, in addition, holomorphicity implies $D_m T^* = 0$, we have that the F^1 and F^2 equations of motion read

$$\begin{aligned} D_n \left(i g^{m\bar{n}} F^1_{m\bar{n}}\right) &= -e D_n (T T^*) \\ D_n \left(i g^{m\bar{n}} F^2_{m\bar{n}}\right) &= e D_n (T^* T)\,, \end{aligned} \tag{2.15}$$

which is just the derivative D_n of the vortex equations (2.12).

Now we turn to the tachyon equation of motion. We have, commuting the D derivatives:

$$\begin{aligned} D^2 T &= g^{m\bar{n}} \left(D_m D_{\bar{n}} T + D_{\bar{n}} D_m T\right) \\ &= g^{m\bar{n}} \left(2 D_m D_{\bar{n}} T - i e F^1_{m\bar{n}} T + i e T F^2_{m\bar{n}}\right) \\ &= -e \left(i g^{m\bar{n}} F^1_{m\bar{n}}\right) T + e T \left(i g^{m\bar{n}} F^2_{m\bar{n}}\right), \end{aligned} \tag{2.16}$$

where in the last line we use the holomorphic properties of T. Thus the tachyon equation of motion now reads

$$\left(i g^{m\bar{n}} F^1_{m\bar{n}}\right) T - T \left(i g^{m\bar{n}} F^2_{m\bar{n}}\right) = -\frac{2\lambda}{e} \left(T T^* T - \alpha^2 T\right). \tag{2.17}$$

This corresponds to pre- and post-multiplying the vortex equations by T and taking the difference. However, this requires that the parameters in the action are related so that

$$\lambda = e^2\,. \tag{2.18}$$

[2] See [27] for more details.

We also get a relation between τ_1, τ_2 and α

$$e\alpha^2 = \pi\left(\tau_1 - \tau_2\right) . \tag{2.19}$$

Note that the relation between λ and e, together with the assumption that the height of the tachyon potential is the tension of the brane system $\mathcal{T}_p$, imply that the tachyon charge e is related to the value of the tachyon field at the minimum of its potential by

$$e^2 \sim \mathcal{T}_p \left(\frac{1}{n_1} \operatorname{Tr}(TT^*)\big|_{\text{minimum}} \right)^{-2} . \tag{2.20}$$

That such a relation holds in general can be established by using the vortex equations to construct the known BPS brane solutions at the minimum of the tachyon potential, as we will do later. It also requires that the trace structure of the potential has a particular form, in particular the quartic terms are $\mathrm{Tr}TT^*TT^*$ with no $(\mathrm{Tr}TT^*)^2$ term [4]. Note that requiring the action to be formulated in terms of a superconnection following Quillen [36, 3, 37], also gives $\lambda = e^2$ and the given trace structure in the potential. In what follows we set $\lambda = e = 1$ for convenience.

If we add that vortex equations, take a trace and integrate over X, we find

$$\tau_1 n_1 + \tau_2 n_2 = \deg E_1 + \deg E_2 \,, \tag{2.21}$$

where the degree of a vector bundle $\deg E$ is defined as

$$\deg E = \frac{1}{V(d-1)!} \int c_1(E) \wedge J^{d-1} \,, \tag{2.22}$$

with J the Kähler form on X and $c_1(E)$ is the first Chern class. Thus we see that τ_1 and τ_2 are completely determined by the parameter α and the bundles E_1 and E_2. In particular

$$\begin{aligned} 2\pi\tau_1 &= 2\pi \frac{\deg E_1 + \deg E_2}{n_1 + n_2} + \frac{2n_2}{n_1+n_2}\alpha^2 \\ 2\pi\tau_2 &= 2\pi \frac{\deg E_1 + \deg E_2}{n_1 + n_2} - \frac{2n_1}{n_1+n_2}\alpha^2 \,. \end{aligned} \tag{2.23}$$

We expect that the solutions of the vortex equations are supersymmetric BPS states. One way to establish this is to analyze the supersymmetry directly. Another way, which we will follow, is to show that these solution satisfy the Bogomol'nyi bound. Separating into holomorphic and anti-holomorphic indices, integrating by parts on the tachyon kinetic terms, commuting the derivatives, and using the identity

$$\int_X \sqrt{g} \operatorname{Tr} F_{m\bar{n}} F^{m\bar{n}} = \int_X \sqrt{g} \operatorname{Tr}\left(i g^{m\bar{n}} F_{m\bar{n}}\right)^2 + \frac{1}{(d-2)!} \int_X \operatorname{Tr} F \wedge F \wedge J^{d-2} \,, \tag{2.24}$$

one can rewrite the action as

$$
\begin{aligned}
S = \int_X \Bigg[& \frac{1}{2} \operatorname{Tr}_1 F^1_{mn} F^{1\,mn} + \frac{1}{2} \operatorname{Tr}_2 F^2_{mn} F^{2\,mn} + 2 g^{m\bar{n}} \operatorname{Tr}_1 D_{\bar{n}} T D_m T^* + \\
& + \frac{1}{2} \operatorname{Tr}_1 \left(i g^{m\bar{n}} F^1_{m\bar{n}} + e T T^* - 2\pi\tau_1 I_1 \right)^2 + \frac{1}{2} \operatorname{Tr}_2 \left(i g^{m\bar{n}} F^2_{m\bar{n}} - e T^* T - 2\pi\tau_2 I_2 \right)^2 \Bigg] + \\
& + \text{topological terms}\,. \qquad (2.25)
\end{aligned}
$$

The topological terms involve the degrees of E_1 and E_2 as well as the second Chern characters

$$
\operatorname{Ch}_2(E_i, J) = \frac{1}{V(d-2)!} \int_X \operatorname{ch}_2(E_i) \wedge J^{d-2}\,, \qquad (2.26)
$$

all of which are fixed once the E_1 and E_2 bundles are chosen. Note that each non-topological term in (2.25) is positive or zero, so that the action is bounded from below by the value of the topological terms. Comparing with equations (2.11) and (2.12), we see that the bound is saturated if and only if one satisfies the vortex equations. Since X is Euclidean, the action is the energy of the configuration. Thus solutions of the vortex equations give the states of global minimum mass for a given set of charges (fixed by the bundles E_1 and E_2). Since this bound is saturated only by supersymmetric BPS states, this implies that solutions of the vortex equations are indeed BPS states.[3]

We have shown that with the relations (2.18) and (2.23) we have an interesting correspondence. Brane configurations where all fields are holomorphic and that arise via the process of tachyon condensation are described by solutions to the vortex equations (2.11) and (2.12). There is one dimensionful parameter, α^2, in the equations which is related to the value of the tachyon at the minimum of the potential and so scales as the string scale. We should emphasize that the holomorphicity conditions (2.11) limit our discussion to tachyon condensation that leads to BPS branes. In order to study stable non-BPS branes we would have to relax these conditions.

2.3 Stable triples

A particularly useful property of the vortex equations is that their solutions are in one-to-one correspondence with a topological notion of stability of the triple (E_1, E_2, T). Thus we can use stability to analyze the existence of solutions on a general X, rather than looking for solutions explicitly.

The analogy here is to the hermitian Yang-Mills equations (HYM) describing the supersymmetric compactification of a single gauge bundle E on a Calabi-Yau

[3]We consider the brane-antibrane worldvolume theory as a system with spontaneously broken supersymmetry.

manifold. They read

$$F_{mn} = F_{\bar{m}\bar{n}} = 0 \qquad ig^{m\bar{n}} F_{m\bar{n}} = 2\pi\tau \,, \tag{2.27}$$

which are a simple subset of the vortex equations. By the Donaldson-Uhlenbeck-Yau theorem, solutions of the HYM equations are in one-to-one correspondence with holomorphic vector bundles of a particular type: those that are poly-stable. This is defined as follows. Let the slope of a bundle be given by

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E} \,. \tag{2.28}$$

A bundle E is stable if for any non-trivial sub-bundle $E' \subset E$, one has $\mu(E') < \mu(E)$. Poly-stability means that E is the direct sum of stable bundles each with the same slope. By this theorem, an analytic problem — solutions of the HYM equations — is a equivalent to a topological problem — listing the stable bundles — and the latter problem is generally much easier to solve. We should note that the stability problem is not quite topological: it also depends on the choice of Kähler form J via the definition of $\deg E$.

It turns out that there is an analogous notion of stability for a triple (E_1, E_2, T), such that there is a solution to the vortex equations if and only if the triple is stable [28]. One first needs to define what is meant by a sub-triple. We take the definition that (E_1', E_2', T') is a sub-triple if

(1) E_i' is a coherent sub-sheaf of E_i with $i = 1, 2$,

(2) T' is the restriction of T.

Equivalently one requires that the diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{T} & E_1 \\ \uparrow & & \uparrow \\ E_2' & \xrightarrow{T'} & E_1' \end{array} \tag{2.29}$$

is commutative. Next one needs the analog of the μ-slope $\mu(E)$. With σ a real number, one defines the σ-slope of a triple (E_1, E_2, T) by

$$\mu_\sigma(T) = \frac{\deg(E_1 \oplus E_2) + \sigma n_2}{n_1 + n_2} \,. \tag{2.30}$$

A triple is then called σ-stable if for all nontrivial sub-triples (E_1', E_2', T') we have

$$\mu_\sigma(T') < \mu_\sigma(T) \,. \tag{2.31}$$

The relation between solutions to the vortex equations (2.11) and (2.12), and the σ-stability of the triple is for $\sigma = \tau_1 - \tau_2$. As seen, for instance, from (2.23), for the brane-antibrane system it means

$$\sigma \equiv \frac{\alpha^2}{\pi} . \tag{2.32}$$

The BPS branes that arise upon the condensation of the tachyon will be σ-stable with σ given by the relation (2.32). As will be very relevant later, since α goes like the string scale M_s, the large volume limit corresponds to large σ. This is the regime where we can trust the vortex equations to provide an adequate description. Note, in particular in the large σ limit, the stability condition (2.31) reads

$$n_2 n_1' - n_2' n_1 > 0 , \tag{2.33}$$

where $n_i' = \operatorname{rank} E_i'$, for $i = 1, 2$, which is similar to a stability condition on a quiver in the orbifold limit [22].[4]

2.4 An example

As a simple illustration of solutions to the vortex equations and some of the discussion to follow, let us consider how we can realize a D0-brane on $\mathbb{C}$ via the condensation of a D2-brane and an anti-D2-brane.

The D2-branes will be realized as U(1) bundles. For finite energy, we require that the connection is pure gauge at infinity (thus we are effectively considering bundles on $S^2 = \mathbb{P}^1$). To ensure that we have a zero brane we need the difference of the bundle charges (2.5) to be one:

$$c_1(E_1) - c_1(E_2) = 1 . \tag{2.34}$$

For simplicity we can take E_2 to be trivial, while E_1 has $c_1(E_1) = 1$.[5] This means that E_1 has a non-trivial holonomy at infinity.

Now consider a solution of the vortex equations (2.11). First T must be holomorphic. This implies that

$$\bar\partial T + i A^1_{\bar z} T - i A^2_{\bar z} T = 0 . \tag{2.35}$$

Equation (2.35) can be solved and gives

$$\begin{aligned} A^1_z - A^2_z &= -i\partial \ln T^* \\ &= -\partial\chi - i\partial \ln f , \end{aligned} \tag{2.36}$$

[4] We thank M. Douglas for valuable discussions on the large σ limit, and for pointing out the relation of (2.33) to the one that arises in the quiver analysis (see [24, appendix A]).

[5] We expect the existence of some equivalence relation between triples, such that different choices of bundles satisfying (2.34) will lead to the same BPS state.

where we have written the tachyon as $T = fe^{i\chi}$. Note that locally gauge transformations can always remove χ. Since fields must be pure gauge at infinity, we have $\ln f \to$ const at infinity. The fact that E_2 is trivial and E_1 has $c_1(E_1) = 1$ means that A_2 can be gauged to zero at infinity while A_1 can be gauged to the form

$$A_1|_\infty = \partial\theta\,, \tag{2.37}$$

where $z = re^{i\theta}$, which is locally trivial but has global holonomy.

From this we see that we can gauge T such that

$$\chi|_\infty = -\theta\,. \tag{2.38}$$

There must be some point $z = z_0$ in $\mathbb{C}$ where this non-trivial holonomy in T untwists, at which point $T = 0$. We can choose this to be the origin $z = 0$. In general this means we can globally gauge T to the form

$$T = f(r)e^{i\theta} \tag{2.39}$$

such that $f(0) = 0$.

The form of f can be determined by subtracting the vortex equations (2.12). We have

$$\begin{aligned} ig^{z\bar{z}}F^1_{z\bar{z}} - ig^{z\bar{z}}F^2_{z\bar{z}} &= i\partial\left(A^1_{\bar{z}} - A^2_{\bar{z}}\right) - i\bar{\partial}\left(A^1_z - A^2_z\right) \\ &= -\partial\bar{\partial}\ln f^2\,, \end{aligned} \tag{2.40}$$

so the difference of the vortex equations (2.12) reads

$$-\partial\bar{\partial}\ln f^2 + 2f^2 = 2\pi(\tau_1 - \tau_2) = 2\alpha^2\,. \tag{2.41}$$

Finding the solution of this equation such that $f(0) = 0$ and $f(\infty) = \alpha$ then completely determines the vortex solution up to gauge transformations. In particular, the individual vortex equations give

$$\begin{aligned} F^1_{z\bar{z}} &= i\left(f^2 - \alpha^2\right) \\ F^2_{z\bar{z}} &= -i\left(f^2 - \alpha^2\right) \end{aligned} \tag{2.42}$$

for the two field strengths.

We see that at infinity $|T| = \alpha$ and the tachyon is at the minimum of its potential. However, as one moves around the S^1 at infinity the phase of the tachyon rotates giving a vortex. The vortex untwists at the origin where $T = 0$. This is the position of the D0-brane. In general there is a modulus to move the D0-brane to any point in the complex plane.

We can do the same analysis without referring directly to the vortex equations. On $\mathbb{P}^1$ we have $E_1 = \mathcal{O}(1)$ and $E_2 = \mathcal{O}$ the trivial bundle. Thus we have the triple

$$\mathcal{O} \xrightarrow{T} \mathcal{O}(1)\,. \tag{2.43}$$

We can think of this as a map between holomorphic functions on $\mathbb{P}^1$ (that is constant functions) to meromorphic functions with, at most, a single pole (at the zero-brane). However, such maps lie in an exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{T} \mathcal{O}(1) \longrightarrow \mathcal{O}_p \longrightarrow 0 \,. \tag{2.44}$$

That is the kernel of T is zero. The cokernel, meanwhile, is simply the sheaf of functions localized at a point $\mathcal{O}_p$. This is precisely the set of points where T vanishes. But since $\mathcal{O}_p$ is localized on a point p it is precisely the description of a D0-brane on p. Thus after condensation, E_1 and E_2 are effectively replaced by their cokernel, representing a D0-brane. Depending on the particular choice of the map T, the D0-brane lies at different points p in $\mathbb{P}^1$.

It is straightforward to show that the triple (2.43) is σ-stable in the sense of (2.30) and (2.31), since any sub-triple has E_2' zero.

2.5 General examples of BPS branes

We can naturally generalize the previous example to construct, via the process of tachyon condensation, supersymmetric $(2d-2)$-branes on a d complex dimensional Calabi-Yau manifold X. Such branes are described by sheaves localized on a holomorphic hypersurface C in X [38, 39, 40, 29]. As above let us assume that E_2 is the trivial U(1) bundle $\mathcal{O}_X$. We then require $c_1(E_1) = [C]$, the class of C. This can be achieved by taking E_1 to be the bundle $\mathcal{O}_X(C)$. Note that in general this bundle also induces lower-dimensional brane charges. Then, for any map T we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{T} \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0\,, \tag{2.45}$$

where $\mathcal{O}_C(C)$ is a sheaf localized on C. As in the previous example, this means we can replace the triple $E_2 \xrightarrow{T} E_1$ with the sheaf $\mathcal{O}_C(C)$. Since this represents a bundle localized on C, it describes a supersymmetric $(2d-2)$-brane configuration as required.

Again, it is easy to see that the triple is σ-stable since any sub-triple has E_2' zero.

3. BPS bound states as stable triples

In this section we will discuss and illustrate the description of BPS branes as stable triples in relation with other existing descriptions of BPS branes.

3.1 Stable sheaves

Let us now relate our description of BPS D-branes as stable triples to existing descriptions of BPS states. Locally, the bosonic D-brane degrees of freedom are a bundle E with gauge fields A_m on the brane together with scalars Φ living in the normal bundle to the brane world-volume C and the adjoint of E. If the brane is an embedding $C \subset X$, then the normal bundle is generically non-trivial, and the scalars are its sections. It has been argued that, the conditions that the background on the brane preserves supersymmetry then lead to a set of first-order holomorphic differential equations on A_m and Φ [39, 29]. For instance, for a brane wrapping a holomorphic two-cycle in K3, one has the Hitchin equations

$$\begin{aligned} F_{z\bar{z}} &= [\Phi_z, \Phi_{\bar{z}}]\,, \\ \bar{D}_{\bar{z}}\Phi_z &= D_z\bar{\Phi}_{\bar{z}}\,. \end{aligned} \tag{3.1}$$

These conditions can be reinterpreted geometrically as a generalized stability condition on the pair (E, Φ).

A second description of a D-brane is as a sheaf $\mathcal{S}$ on X. In particular, it was conjectured in [29] that BPS branes correspond to "coherent semistable" sheaves on X. The coherent condition means that $\mathcal{S}$ fits into an exact sequence

$$E_2 \longrightarrow E_1 \longrightarrow \mathcal{S} \longrightarrow 0\,, \tag{3.2}$$

where E_1 and E_2 are vector bundles (or more precisely, the sheaves of sections of vector bundles). The semi-stability condition is the generalization to sheaves of the geometrical condition of stability for vector bundles. However, in contrast to the case for vector bundles, there is no differential equation on the sheaf corresponding to the condition of stability. From this point of view, the requirement that the sheaves are stable is a conjecture.

The sheaf description is related to the description in terms of fields on the embedded brane C as follows. One requires that on C the sheaf $\mathcal{S}$ reduces to the vector bundle E. Furthermore, away from C the sheaf must be zero. Mathematically this means that the support of $\mathcal{S}$ is C, and the restriction of $\mathcal{S}$ to C is E,

$$\mathrm{supp}(\mathcal{S}) = C\,, \qquad \mathcal{S}|_C = E\,. \tag{3.3}$$

In fact, these conditions do not completely determine $\mathcal{S}$. One can show that the additional information required is precisely the twisting of the scalar fields Φ. Thus, locally, $\mathcal{S}$ is equivalent to the pair (E, Φ). In the case of C being a curve in K3 one can then see a close relation [41] between stable sheaves $\mathcal{S}$ and solutions of the local Hitchin equations (3.1). However, in general, there is no explicit justification of the requirement that $\mathcal{S}$ be stable.

3.2 Stable triples

We are proposing a description of BPS brane states as stable triples. How does this relate to the sheaf and local-field descriptions? Consider the large volume limit where we can neglect stringy corrections. In this limit we can derive an important result:

In the large σ limit, any stable triple will necessarily be without a kernel, that is, the map T will be injective (one-to-one).

To see this, suppose there is a kernel, $\ker(T) \subset E_2$. By definition, there is then a non-trivial sub-triple $T' : \ker(T) \to 0$. Since E_2 is torsion-free, any sub-sheaf of E_2 must be torsion-free. This implies that if $\ker(T)$ is non-trivial, it must be supported on the whole of X. In particular, we must have $\mathrm{rank}(\ker(T)) > 0$. Recall that for large σ the stability condition reduced to a condition on the ranks (2.33). However, for the sub-triple $T' : \ker(T) \to 0$ we have

$$n_2 n_1' - n_1 n_2' = -n_1 n_2' < 0\,, \tag{3.4}$$

since $n_1' = 0$, $n_2' > 0$ and so the stability condition (2.33) is violated. Thus any stable triple has $\ker(T) = 0$. In general, it will have a cokernel however. The fact that $\ker(T) = 0$ implies that there is always an exact sequence

$$0 \longrightarrow E_2 \xrightarrow{T} E_1 \longrightarrow \mathrm{coker}(T) \longrightarrow 0\,. \tag{3.5}$$

Comparing with (3.2), we see that, in the large σ limit, the coherent sheaf $\mathcal{S}$ is simply the cokernel $\mathrm{coker}(T)$ of the tachyon map. In particular, it will be supported on some holomorphic subspace of X (or X itself). For example on K3, if E_1 and E_2 of the same rank, then the brane charge $c_1(E_1) - c_1(E_2) = c_1(\mathcal{S})$ must be effective, i.e. the BPS branes are realized as a sheaf $\mathrm{coker}(T)$ localized on a holomorphic curve C. In particular, $c_1(E_1) - c_1(E_2) = n[C]$, where n is the rank of $\mathrm{coker}(T)$ on C.

This appears to justify the conjecture that D-branes are described by stable coherent sheaves. However, it turns out that the notion of stability for an injective triple (with T injective), is different from the notion of stability of the cokernel $\mathcal{S} = \mathrm{coker}(T)$, considered either as a torsion sheaf on X or as a vector bundle on its support. The reason being that the σ-slope (2.30) in the σ-stability of the triple (2.31) involves only the ranks and the first Chern classes of E_1 and E_2. On the other hand, the μ-slope (2.28) in the μ-stability of the cokernel $\mathcal{S}$ involves its first Chern class. The latter is related to the second Chern classes of E_1 and E_2, which do not enter the σ-slope stability.

One might expect that the vortex equations receive corrections involving higher-order Chern classes, which could correct this discrepancy. However, in fact, there is a basic difference between the stability of the triple and other notions of D-branes stability. In general, one considers the charges of a brane-antibrane system as elements in K-theory, and searches for geometric objects that correspond to these charges. In this paper, we suggest a representation of the brane-antibrane charges by holomorphic triples, which satisfy the vortex equations. For an injective triple, the K-theory charges are then related to the Chern classes of $\mathrm{coker}(T)$, the difference of $\mathrm{ch}(E_1)$ and $\mathrm{ch}(E_2)$. However, this proposal implies that the stability condition does not depend only on the K-theory classes of a complex and its sub-complexes, but rather on the individual terms.[6] The σ-slope is expressed in terms of the sums of the ranks and degrees of the individual terms in the complex, rather than their alternating differences.

As we will discuss in the next section, we can also consider more general webs of vector bundles. Basically we are working with the representation of some quiver in the category of vector bundles on X. We fix the 'shape' of the web of bundles and maps and then look for stability of all configurations of bundles and maps for that shape. Thus, for instance, for vector bundles we take the quiver of type A_1, for stable triples we take the quiver A_2 and for a sequence of n vector bundles we will take the quiver A_n. For all such objects one can define stability, but there are many notions of slope and stability (depending on discrete and continuous parameters) [42, 43].[7] All of these stability notions specialize to the ordinary slope stability of vector bundles when working with A_1. However already for A_2 there are many different stabilities. For example there is one continuous family of stabilities (depending on σ or τ) that we use.

Finally, note that since there are corrections to the effective action (2.7), we expect the vortex equations to be deformed. This deformation is likely to influence the stability notion when the corrections are not negligible, and in particular in the finite σ regime.

3.3 Non-injective tachyons

Away from the large σ limit, an interesting new possibility arises: there may be stable triples with non-injective T. While we might expect corrections in this regime, let us, for now neglect them and discuss this possibility in more detail.

Suppose first that $n_1 = n_2 = 1$, i.e. E_1 and E_2 are global line bundles on X. In this case there are two possibilities: either T is the zero map or T is injective. Thus in this case no new states appear. To show this note, as above, that that $\ker(T)$ must be a sub-sheaf in E_2 and in particular is torsion free, since E_2 has no torsion. Its rank is smaller or equal to $n_2 = 1$. Now, if $\mathrm{rank}(\ker(T)) = 0$, then $\ker(T) = 0$ since it cannot be supported on any sub-variety. If $\mathrm{rank}(\ker(T)) = 1$, then $\ker(T) = E_2$ since in this case we will have $E_2/\ker(T) \subset E_1$ is supported on a sub-variety and hence must be zero since E_1 has no torsion.

Consider next a more general case, let $n_1 = n_2 \neq 1$. In this case we can have solutions to the vortex equations, i.e. stable triples, with a non-injective tachyon. As an example, consider the following. Let $X \subset \mathbb{P}^3$ be a general quartic K3. Then the Picard group $\mathrm{Pic}(X) = \mathbb{Z}$ is generated by the hyperplane line bundle $h = \mathcal{O}_{\mathbb{P}^3|X}(1)$. The tangent bundle T_X of X is h-stable. In particular all line sub-bundles of T_X will

[6]We thank M. Douglas for this comment.

[7]For the application of [42] to the stability analysis in orbifold points see [22, section 7].

be of the form h^{-a} for some positive integer a. Let a be the minimal positive integer, for which $h^{-a} \subset T_X$. Then we have a short exact sequence on X:

$$0 \longrightarrow h^{-a} \longrightarrow T_X \longrightarrow h^a \otimes I_Z \longrightarrow 0\,,$$

where $Z \subset X$ is a finite set consisting of $24 + 4a^2$ points and I_Z is the ideal sheaf of Z. Since $I_Z \subset \mathcal{O}_X$, we get a natural inclusion $h^a \otimes I_Z \subset h^a$ and so we get a composition map

$$T : T_X \to h^a \otimes I_Z \subset h^a \subset T_X \otimes h^{2a}\,.$$

Thus, in this example $E_2 = T_X, E_1 = T_X \otimes h^{2a}$ and $n_1 = n_2 = 2$.

By definition $\ker(T) = h^{-a} \neq 0$. We have to check that the triple $(T_X \otimes h^{2a}, T_X, T)$ is σ-stable for some choice of σ. Recall the σ-stability condition (2.30), (2.31). We will work with the polarization h on X. The corresponding slopes of the members of our triple are:

$$\begin{aligned} \mu(T_X) &= \frac{c_1(T_X)\cdot h}{2} = 0\,, \\ \mu(T_X \otimes h^{2a}) &= \frac{(c_1(T_X \otimes h^{2a})\cdot h}{2} = 2ah^2 = 8a\,, \end{aligned} \tag{3.6}$$

since $h \cdot h = 4$, and so

$$\mu_\sigma(T) = \frac{16a + 2\sigma}{4}\,. \tag{3.7}$$

There is a lower bound on σ which is universal for any stable triple (see [28, Proposition 3.13]):

$$\begin{aligned} &\sigma > \mu(E_1) - \mu(E_2)\,, \\ &\sigma > 0\,. \end{aligned} \tag{3.8}$$

Here it means $\sigma > 8a$.

To test for stability we need to consider a sub-triples (E_1', E_2', T') such that $E_2' \subset T_X$, $E_1' \subset T_X \otimes h^{2a}$ are (saturated) subs-heaves and $T(E_2') \subset E_1'$. The cases when $E_2' = 0$ is obvious. Let $E_2' = T_X$, there is then only one non-trivial case to consider:

(1) $T' : T_X \to h^a \otimes I_Z$. We need to check that $\mu_\sigma(T') < \mu_\sigma(T)$. We have

$$\mu_\sigma(T') = \frac{4a + 2\sigma}{3}\,. \tag{3.9}$$

Comparing (3.7) and (3.9) we get the stability condition $\sigma < 16a$.

Consider next E_2' of rank one. The saturation condition then implies that E_2' is a line bundle and so $E_2' = h^{-k}$ with $k \geq a$. We have

$$\mu_\sigma(T') = \frac{-4k + c_1(E_1')\cdot h + \sigma}{1 + \mathrm{rank}(E_1')}\,. \tag{3.10}$$

There are three cases to consider:

(2) $E_1' = 0$. For this to be a sub-triple we need that $E_2' \subset \ker(T) = h^{-a}$, which is satisfied since any line bundle h^{-k} maps non-trivially as a sub-sheaf in h^{-a}, as long as $k \geq a$. In this case, $\mu_\sigma(T') = -4k + \sigma$ and so we must require $\sigma < 16a$.

(3) E_1' has rank one and so is a line bundle. Then $E_1' = h^m$ with $m \leq a$ and so

$$\mu_\sigma(T') = \frac{-4k + 4m + \sigma}{2} \leq \frac{-4a + 4a + \sigma}{2} = \frac{1}{2}\sigma\,, \tag{3.11}$$

which, by equation (3.7) is always strictly smaller than $\mu_\sigma(T)$.

(4) $E_1' = T_X \otimes h^{2a}$. In this case

$$\mu_\sigma(T') = \frac{-4k + 16a + \sigma}{3} \leq 4a + \frac{1}{3}\sigma\,, \tag{3.12}$$

which is again smaller than $\mu_\sigma(T) = 8a + \sigma/2$.

Therefore, in conclusion, the triple $(T_X \otimes h^{2a}, T_X, T)$ will be σ-stable as long as

$$8a < \sigma < 16a\,. \tag{3.13}$$

Let us discuss now the large σ limit. From the above we see that when $\sigma > 16a$ the sub-triples $T' : T_X \to h^a \otimes I_Z$ and $T' : h^{-k} \to 0$ have a larger σ-slope than that of the original triple. Therefore they will destabilize the triple in the large volume limit. Physically it means that the vortex solution corresponding to the triple will decay in the large volume limit.

Finally, note that if we consider the case $n_1 \neq n_2$ then there is an upper bound on σ for any stable triple (see [28, Proposition 3.14]):

$$\sigma < \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)(\mu(E_1) - \mu(E_2))\,. \tag{3.14}$$

It implies that there are no stable triples with $n_1 \neq n_2$ in the large σ limit. At first glance, this may appear a little odd, since one could certainly take E_1 and E_2 to be trivial, which should lead, after tachyon condensation, simply to $n_1 - n_2$ D-branes wrapping X. However, the point is that one expects such a configuration to appear as a semi-stable rather than stable configuration, since the n_2 brane-antibrane pairs which annihilate decouple from the surviving $n_1 - n_2$ branes.

4. Generalizations and dimensional reduction

In this section we discuss some generalizations of the brane-antibrane system. We will also show that the theory of stable triples suggests some curious relations between brane-antibrane configurations and wrapped branes in higher dimensions.

4.1 Other brane-antibrane configurations

There is a natural generalization of the previous discussions when we allow the E_1 and E_2 bundles to split [31]. Suppose the $\mathrm{U}(n_1)$ gauge group for E_1 is split into $\bigoplus_i \mathrm{U}(n_{1,i})$ and similarly the gauge group for E_2 splits into $\bigoplus_j \mathrm{U}(n_{2,j})$. We can then write the bundles as $E_1 = \bigoplus_i E_{1,i}$ for the branes and $E_2 = \bigoplus_j E_{2,j}$ for the antibranes. The form of the vortex equation generalizes. In particular, thus far, we assumed that the right-hand sides of equations (2.12) were simply proportional to the identity matrix since, in general, this was the only constant matrix in the gauge group for E_1 and E_2. However, if E_1 and E_2 split then we have more possibilities.

The vortex equations generalize to

$$\begin{aligned} ig^{m\bar{n}}F^{1,i}_{m\bar{n}} + e\sum_j T_{i,j}T^*_{i,j} &= 2\pi\tau_{1,i}I_{1,i}\,, \\ ig^{m\bar{n}}F^{2,j}_{m\bar{n}} - e\sum_i T^*_{i,j}T_{i,j} &= 2\pi\tau_{2,j}I_{2,j}\,, \\ \sum_j T_{i,j}T^*_{k,j} &= 0\,, \qquad \text{for } i\neq k, \\ \sum_i T^*_{i,j}T_{i,k} &= 0\,, \qquad \text{for } j\neq k. \end{aligned} \tag{4.1}$$

Here $F^{1,i}$ and $F^{2,j}$ are the field strengths for the $\mathrm{U}(n_{1,i})$ branes and $\mathrm{U}(n_{2,j})$ anti-branes respectively. Similarly the tachyon, which is a map from E_2 to E_1, splits into a web of maps $T_{i,j}$ between $E_{2,j}$ and $E_{1,i}$. Corresponding to these generalized vortex equations, we would expect there to be a generalized notion of stability for the web of bundles and maps.

Other generalizations are also possible. In the example above we assumed that all the maps $T_{i,j}$ were holomorphic. In general, however, some may be anti-holomorphic. For example, suppose we have equal numbers of $E_{1,i}$ and $E_{2,j}$ and that the tachyon maps are zero except for $T_{i,i}$ which is holomorphic and $T_{i+1,i}$ which is anti-holomorphic. Relabelling, this can be written as the A_n quiver:

$$E_n \xrightarrow{T_{n-1}} E_{n-1} \xrightarrow{T_{n-2}} E_{n-2} \xrightarrow{T_{n-3}} \cdots \xrightarrow{T_1} E_1\,. \tag{4.2}$$

The corresponding vortex equations are then given by

$$\begin{aligned} ig^{m\bar{n}}F^{i}_{m\bar{n}} + eT_iT^*_i - eT^*_{i-1}T_{i-1} &= 2\pi\tau_i I_i\,, \\ T_{i-1}T_i &= 0\,, \end{aligned} \tag{4.3}$$

where F^i is the field strength for the bundle E_i. The second condition on the maps implies that the sequence forms a complex. The relationship between the stability condition for the configuration (4.2) and the the solutions of the first set of vortex equations (4.3) has been worked out in detail in the recent paper [45].

In general, one can consider an arbitrary quiver such that there is a Z_2-grading of nodes into branes and anti-branes and a set of holomorphic maps which always connect brane nodes and anti-brane nodes. The resulting stability and vortex equations have not been explored in the mathematics literature for quivers, which are not of type A_n.

4.2 Dimensional reduction and stable triples

In the following we we will discuss an interesting relation between the triple (E_1, E_2, T) on X and a single vector bundle on $X \times \mathbb{P}^1$ [27, 28] and its interpretation.

Denote the two natural projections from $X \times \mathbb{P}^1$ to X and $\mathbb{P}^1$ by

$$p : X \times \mathbb{P}^1 \to X\,, \qquad q : X \times \mathbb{P}^1 \to \mathbb{P}^1\,. \tag{4.4}$$

Consider the holomorphic tangent bundle of $\mathbb{P}^1$. This is naturally invariant under the SU(2) rotations of the sphere $\mathbb{P}^1$. Geometrically it is the bundle $H^{\otimes 2}$ where H is the line bundle over $\mathbb{P}^1$ with Chern class one.

We can now consider making a dimensional reduction for a set of D-branes wrapping the $X \times \mathbb{P}^1$. However, rather than assuming the gauge-field flux is zero, we will assume that on $\mathbb{P}^1$ the field strength is given by the SU(2)-invariant bundle described above. In particular, we will take the bundle on $X \times \mathbb{P}^1$ to have the form

$$F = p^* E_1 \oplus \left(p^* E_2 \otimes q^* H^{\otimes 2}\right)\,, \tag{4.5}$$

where E_1 and E_2 are arbitrary bundles on X. Since $\operatorname{rank} F = n_1 + n_2$, we have (n_1+n_2) $(2d+2)$-branes, while $c_1(H^{\otimes 2}) = 2$ implies that such a bundle also describes $2n_2$ $2d$-branes wrapping X.

Without a non-trivial bundle on $\mathbb{P}^1$, there would be no scalars in the dimensional reduction, since the are no covariantly constant one-forms on $\mathbb{P}^1$. However, with the particular bundle described above, there is a pair of covariantly constant one-forms. These form a complex scalar in the dimensional reduction. In particular, the gauge connection will have the form

$$A = \begin{pmatrix} A_1 & T\beta \\ T^*\beta^* & A_2 \end{pmatrix}, \tag{4.6}$$

where A_1 is the E_1-connection on X, A_2 the E_2-connection on X and β and β^* are the covariantly constant one-forms on $\mathbb{P}^1$. The one-forms can be normalized such that $\beta \wedge \beta^*$ is the Kähler form on $\mathbb{P}^1$. The field T and its conjugate T^* are in the bifundamental of E_1 and E_2.

Performing the dimensional reduction of the Yang-Mills action for F on $\mathbb{P}^1 \times X$ we find precisely the action for F_1 and F_2 with tachyon T as given in (2.7), with the parameter α^2 (2.8) given by the inverse volume of $\mathbb{P}^1$,

$$\alpha^2 = \frac{2\pi^2}{V_{\mathbb{P}^1}}, \tag{4.7}$$

where $V_{\mathbb{P}^1}$ is the volume of the $\mathbb{P}^1$. Thus, the theory of branes and anti-branes wrapped on X is obtained as a particular SU(2)-invariant reduction of D-branes on $X \times \mathbb{P}^1$. Furthermore, it can be shown [28] that the conditions of a stable triple (E_1, E_2, T) are equivalent to requiring the bundle F to be stable. While the brane-antibrane configuration in general is not supersymmetric, the brane on $X \times \mathbb{P}^1$ can be supersymmetric.[8] In general, however, it may be that there is no supersymmetric bound state with all the relevant brane charges. The dimensional reduction relation implies that when the brane-antibrane configuration is non-supersymmetric, there is no supersymmetric bound state with the corresponding brane charges on $X \times \mathbb{P}^1$.

There is one further generalization. It has been shown in [44] that general SU(2)-invariant holomorphic bundles over $X \times \mathbb{P}^1$ are in one-to-one correspondence with a set of bundles and maps (4.2). There is a corresponding generalized notion of stability. One would expect that this coincides with the generalized vortex equations (4.3) discussed in the previous section. A mathematically rigorous analysis of this correspondence is carried out in the very interesting recent paper [45].

Finally, note that from (4.7) we see again that since α^2 goes like the volume of the 2-sphere (which is of the string size) in the dimensional reduction, the region of validity of the vortex equations is at large $\sigma \sim \alpha^2 \sim M_s^2$.

Acknowledgments

We would like to thank M. Douglas for valuable discussions. Y. Oz would like to thank Tel-Aviv university for hospitality during part of this work. T. Pantev is supported in part by an NSF grant DMS-9800790 and by an Alfred P. Sloan Research Fellowship. D. Waldram would like to thank The Enrico Fermi Institute at The University of Chicago and the Physics Department of The Rockefeller University for hospitality during the completion of this work.

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